Many low-energy properties of condensed matter systems can be understood with the help of topological invariants. These are integer numbers related to certain abstract features of quantum states, which can acquire non-vanishing phase, called Berry phase, under parallel translation along a closed loop in a parameter space (of momentum, position or other). They explain the robustness of the accurate quantization of physical observables such as the relationship between the frequency and voltage in the ac Josephson effect, the magnetic flux quantization in superconductors and the quantization of the conductance in the quantum Hall effect. The most important property of topological states is the closing of the energy gap at the boundary between phases with different values of the topological invariant and the so-called bulk-boundary correspondence saying that on the surface of an insulator with a given topological invariant there are always metallic states, i.e. with a zero gap, whose number is equal to this invariant. Topological boundary states are particularly interesting because of their possible applications, among others in quantum computing, spintronics and metrology. The edge states are, for example, Majorana states, which are characterized by exotic quantum statistics, other than fermions and bosons, which give hope for topologically protected qubits. The recent research on topological materials has concentrated on symmetry-protected topological invariants, which arise due to momentum-space topological defects and describe the existence of protected surface states. The main advantage in the topological considerations is that the topological invariants take discrete values and therefore they cannot be modified by weak perturbations. This topological protection is guaranteed as long as the symmetries of the system, which allow the definition of the various topological invariants, are obeyed.

However, experimental studies have led to significant discrepancies with the simple theoretical models, questioning the whole topological nature of these systems. In particular, there is still no consensus about the interpretation of the experimentally observed lack of topological protection of the edge state transport in the model topological insulator of HgTe/CdTe quantum wells, i.e., inaccuracy of conductance quantization in the absence of an external magnetic field. Breaking away from the popular trend focusing on idealised simple models for these topological insulators we will develop theory for real-life *topological semiconductors* taking into account the actual lattices and rigorous and approximate symmetries of these materials.

One of the most important issues studied within this project will be to solve the problem of quantization of conductivity in quantum wells based on preliminary results that indicate the presence of additional and unexpected edge states in the energy gap. Their number is proportional to the number of HgTe/CdTe layers in the system and they do not result from the standard, simplified model of these wells. The working hypothesis is that electrons in these additional states may be localized due to disorder and be a source of decoherence and dephasing for helical states, which we should typically obtain in a quantum well. Under this project, quantum transport calculations will be carried out to verify this hypothesis. Another direction of research will be designing a quantum well in which no additional boundary states will appear. To this end, it is necessary to analyze topological invariants that show their presence and to understand how they can be controlled, e.g., by adding several layers of another semiconductor in which these invariants would be opposite. Topological semiconductors may also have defects. One of the most interesting are the atomic steps at the surface. Both theory and experiment indicate that another topological states may appear at such steps. They are characterized by high density of states, because their dispersion becomes flat when we increase the number of layers of the system. This can lead to spontaneous breaking of symmetry and formation of ordered states. On one hand, such states can generate further topological states when domain walls occur at the step, which give a similar energy spectrum as Majorana states. On the other hand, a higher number of steps can lead to an effective description like in graphene Moiré superlattices, where extremely interesting magnetic and superconducting states are obtained.

Topological semiconductors can also be studied as open systems, where energy is not preserved and the effective description is given by non-Hermitian Hamiltonians. This happens, for example, when we place such a semiconductor in a photon resonator, where the photons interact with the particle-hole excitations in the semiconductor and so-called exciton-polaritonite condensate is formed. This state may also have a non-trivial topology and an interesting question is whether and how it can be inherited from semiconductor electrons. Currently, it is also possible to create resonator networks where polaritons bands are formed, in analogy to electron bands in a crystal. Such systems may have unique non-Hermitian topological invariants, such as the hidden Chern number that I have discovered in one dimension, and they will be studied as part of this project.